CHAPTER 5

Point Elasticity, Total Revenue, and Marginal Revenue

The discussion in the preceding chapter dealt only with the elasticity of the curve as a whole. But this is rather a rough-and-ready concept, for "the elasticity" of a curve is really a sort of average of the elasticities at different points along the curve. "The elasticity" we have been dealing with is often called *arc* elasticity, the elasticity of the arc of the curve. It is a sort of average elasticity. Whatever term is used (arc or average), it relates to the elasticity of the curve or arc as a whole. We come now to consider *point* elasticity, the elasticity at any point on a curve.

CURVES WITH AVERAGE ELASTICITY OF UNITY

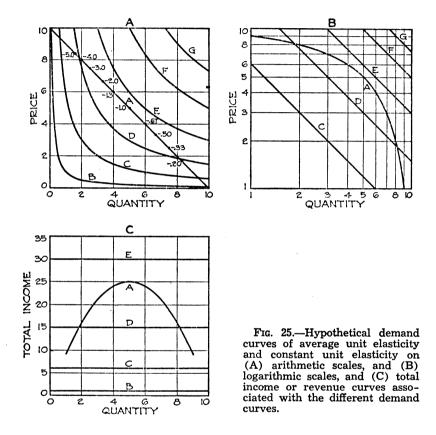
Elasticity is a proportional concept, and the elasticity of a straight line curve on a chart with arithmetic scales therefore is not constant from point to point along the line. It varies from point to point. This shows up most clearly in the simplest case of a straight-line curve with an average or arc elasticity of unity; such a curve is represented by the straight line marked A in section A of Figure 25.

This straight-line curve has an elasticity of unity at its central point where price = 5, and quantity = 5; for if dx = dy, as it does here when the slope of the line is 45°, then $\frac{dx}{-dy} \cdot \frac{5}{5} = -1.0$.

But at other points along the line the elasticity is not -1. At the point where price = 6, and quantity = 4, for example, the elasticity is $\frac{dx}{dy} \cdot \frac{6}{-4} = -1.5$. At the point where price = 8, and quantity = 2, the elasticity is -4.0. Conversely, at points below and to the right of the center of the line, the elasticities are less than -1.

What would a curve of constant unit elasticity at all points look like? It follows from the preceding paragraph that the slope of the line $(\frac{dy}{dx})$ at every point would have to be proportional to the

relation between y and x at every point. A curve of this sort would be a rectangular hyperbola, approaching the x and y axes as asymptotes. Several constant-unit elasticity curves of this sort, lettered B, C, D, etc., are plotted in section A of Figure 25 along with



the straight-line curve A that has an average elasticity of unity.

This figure shows graphically how the elasticity of a straight line changes from point to point. It shows this by comparison of the straight-line curve, A, with the constant elasticity curves, B, C, D, etc., beside it. This comparison shows that the upper part of the straight-line curve is less steeply sloped than the constant unit elasticity curves; that is, it is more elastic than unity. Conversely, the lower part of the straight-line curve is less steeply sloped than unity. The elasticity is highest at the upper end and lowest at the lower

end. It is higher than unity at the top, decreases to unity at the middle, and gets less and less than unity from there on down. The elasticity at different points along the straight line is shown by the series of figures written beside the line.

The situation can be shown on double logarithmic paper, as in section B of Figure 25. The constant unit elasticity curves, B. C, D, etc., shown in section A of Figure 25, become straight lines with slopes of 45° on the logarithmic paper used in section B. The straight-line curve, A, on arithmetic paper in section A undergoes the opposite change to become a curved line, convex from above,¹ on the logarithmic paper used in section B.

TOTAL REVENUE OR INCOME

The total revenue (that is, total income) that would be realized from the sale of different quantities of a commodity depends upon the shape or curvature of the demand curve, as well as upon its elasticity. The total revenue curve is directly related to the demand curve (which is in other words the *average* revenue curve). They can be shown on similar charts, the only difference being that in the total revenue chart the vertical scale shows total revenues instead of average revenues (prices).

The total revenue curve associated with a straight-line demand curve on arithmetic paper, with an average elasticity of unity, is shown in section C in Figure 25. This shows that with this sort of a demand curve the maximum total revenue is realized from an average crop. Large crops and small crops both bring in less money than average crops. The point of highest total revenue comes at the point where the elasticity of the demand curve is unity.

The total revenue curves derived from constant unit elasticity

¹There are two ways of verbally describing the curvature of lines plotted on co-ordinate paper. Both of them are in common use. The one way is to describe curves as concave or convex from above (a basin with water in it is concave from above) while the other way is to describe them as concave or convex to the origin.

There are objections to both systems. The objection to the "from above" reference is that it cannot be applied to curves whose ends lie on the same vertical line. The objection to the "origin" reference is that it cannot be applied to curves that go through the origin. This objection is perhaps more important than the other, because some important economic curves necessarily start from the origin—total revenue curves, positive sloping curves of unit elasticity, etc. In addition, most mathematicians (although not R. G. D. Allen) and some economists use the reference "from above"—Joan Robinson in England, and Tintner, Waugh, and Thomsen in the United States, to name only a few. We are accordingly using the reference "from above" in the present work.

curves like B, C, D, etc., are horizontal straight lines. If the elasticity is unity at all points, the total revenue remains constant, whatever the size of the crop, as shown for the curves B, C, D, and E in section C of Figure 25.

Concave demand curves with an average elasticity of unity, but less curved on arithmetic paper than the constant elasticity curves shown in section A of Figure 25, undergo an interesting transformation when plotted on logarithmic paper. Their curvature is reversed. They are concave on arithmetic paper, but they become convex on logarithmic paper. The point of highest total revenue appears (in their case as in the case of straight-line curves on arithmetic paper) at the central point of the curve where the point elasticity is unity.

Concave demand curves with an average elasticity of unity, but more curved on arithmetic paper than constant-elasticity curves, lose some of their curvature when plotted on logarithmic paper. But they retain their concavity. Accordingly, with this kind of demand curve, the *minimum* total revenue is realized from the sale of an average crop. Large crops and small crops both bring in more money than an average crop. The point where the elasticity of the demand curve is unity is the point of lowest total revenue.

ELASTICITIES OTHER THAN UNITY

Demand curves whose elasticity is constant but higher or lower than unity are straight lines on logarithmic scales, like the curves B, C, D, etc., in section B of Figure 25, but their slopes are other than 45°. The slopes of the inelastic curves are steeper than 45°, and those of the elastic curves, flatter than 45°.

The total revenue curves associated with these constant (but not unit) elasticity demand curves are sloping curved lines, not horizontal straight lines like the total revenue curves associated with constant unit elasticity curves. They are curved lines. The total revenue curves for constant but less than unit elasticity demand curves are rectangular hyperbolas like the curves of constant elasticity shown earlier in section A of Figure 25. And like those curves, they would be straight lines with a negative slope on double logarithmic paper. A demand curve with constant elasticity of -0.5 is shown on arithmetic and logarithmic paper, together with the total income or revenue curve based upon it, in sections A and C of Figure 26.

The total revenue curves for constant but more than unit

elasticity demand curves are also curved, but they are parabolas, with apex at the origin of the x and y axes. They, too, are straight lines on double logarithmic paper; but they have a positive slope.

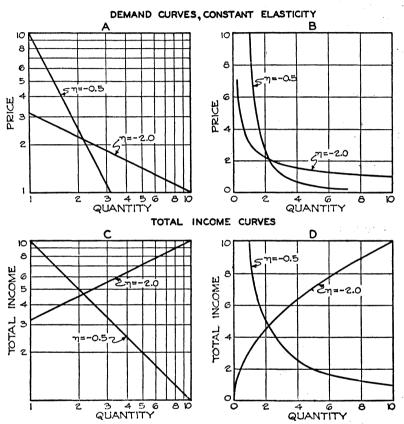


FIG. 26.—(A and B) Hypothetical constant elasticity demand curves and (C and D) total income (revenue) curves associated with them.

A demand curve with a constant elasticity of -2.0, and the total revenue curve based upon it, is shown also in sections A and B of Figure 26.

The elasticity figures given beside the total revenue curves in the lower sections of this chart are only identification-elasticities, showing in each case the elasticity of the demand curve from which the total revenue curve was derived, not the elasticity of the total

Agricultural Price Analysis

revenue curve itself. That elasticity can be figured mathematically from the elasticity of the demand curve. The mathematical relationship between the elasticity of the demand curve and that of its associated total revenue curve is comparatively simple. If E = the elasticity of the total revenue curve, defined in an analogous manner to the Marshallian definition of the elasticity of the demand curve

(i. e., as
$$\frac{dq}{d(pq)} \cdot \frac{pq}{q}$$
 or $\frac{dq}{q} / \frac{d(pq)}{pq}$) and $e =$ the elasticity of

the demand curve, the relationship² is $E = \frac{e}{e+1}$

This formula may be solved for e in terms of E, giving the elasticity of the demand curve in terms of the elasticity of the total revenue curve, as follows: $e = \frac{E}{1-E}$

THE MEASUREMENT OF POINT ELASTICITY

Point elasticity can be measured mathematically by the use of the same formula that was used for average or arc elasticity in the preceding chapter. If the demand curve is a straight line, the

² This relationship may be derived as follows:

$$E = \frac{dq}{q} / \frac{d(pq)}{pq} = 1 / \frac{d(pq)}{pdq} = 1 / \frac{pdq + qdp}{pdq}$$
$$= 1 / \left(1 + \frac{qdp}{qdp}\right) = 1 \quad \left(1 + \frac{1}{e}\right) = \frac{e}{e+1}$$

It is interesting to observe that if E and e are both defined as reciprocals of the usual definitions, that is, as $\frac{d(pq)}{dq} \cdot \frac{q}{pq}$ and $\frac{dp}{dq} \cdot \frac{q}{p}$ respectively, then the relation between E and i is some simple. It is

relation between E and e is very simple. It is

$$E = \frac{d(pq)}{dq} \cdot \frac{q}{pq} = \frac{d(pq)}{dq} \cdot \frac{1}{p} = \frac{qdp + pdq}{pdq} = \frac{qdp}{pdq} + 1 = e + 1.$$

(See R. G. D. Allen, Mathematical Analysis for Economists, Macmillan, London, 1938, p. 252.) That is, E as thus defined is always greater by 1 than e as thus defined.

If e is defined as usual $(as \frac{dq}{dp} \cdot \frac{p}{q})$ and E defined as above $(as \frac{d(pq)}{dq} \cdot \frac{q}{pq})$ then $E = \frac{1}{q} + 1$.

I am indebted to Gerhard Tintner and Adolf Kozlik for this footnote.

formula is merely the original Marshallian formula, with the p and q at the particular point used in place of the *average* p and *average* q over the range used when average elasticity is computed. If the curve is not a straight line, then a tangent must be drawn to it at

the point where the elasticity is to be measured. The $\frac{dq}{dp}$ is then

computed from the tangent.

Point elasticity can also be measured graphically. The way to do it is shown in Figure 27. If the demand curve is a straight line and the elasticity at the point p is to be found, that can be done by laying a ruler along the demand curve and measuring the two distances (1) from pto the point where the ruler cuts the y axis (pc in the diagram) and (2) from p to the point

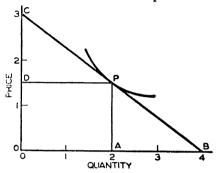


FIG. 27.—Diagram for the measurement of point elasticity.

where the ruler cuts the x axis (pb in the diagram). The latter distance divided by the former (that is, $\frac{pb}{pc}$) then gives the elasticity

at the point p.

If the demand curve is a curved line, the procedure is the same, but the ruler is laid tangent to the curve at the point where the elasticity is to be measured. The ratio $\frac{pb}{pc}$ then gives the elasticity at that point.³ It can then be shown that the elasticity at pis also given by $\frac{ab}{oa}$.

MARGINAL REVENUE

The preceding sections have shown the relation between elasticity and total income or revenue. We turn now to a third concept, marginal revenue.

⁵ The proof of these relations is simple. The elasticity, $\frac{dq}{dp} \cdot \frac{p}{q}$, is the change in quantity divided by the change in price, multiplied by the price divided by

The concept or definition of marginal revenue is perfectly clearcut. The total revenue (or total income, which means the same thing) is the total revenue from the sale of a given amount of the product, say x bushels. It is computed by multiplying x bushels by the price at which that number of bushels can be sold. This total revenue may be compared with the total revenue from the sale of x + 1 bushels; this is computed by multiplying x + 1bushels by the price at which that number of bushels can be sold. The difference between the two total revenues is the marginal revenue.⁴

For example, a dealer may be able to sell 10 boxes of apples a day for \$2.00 a box. His total revenue from the sale of apples then is \$20.00. Suppose now that more apples come on the market; he now has 11 boxes a day to sell. He has to cut the price to move them all. He cannot merely cut the price of the eleventh box; the buyers of the 10 boxes would object; so he has to cut the price of all the boxes of apples. If he has to cut the price to \$1.90 per box, his total revenue then is \$1.90 \times 11, which is \$20.90. What is the marginal revenue, then? It is the difference between \$20.00 and \$20.90; it is 90 cents.

Suppose then that still more apples come on the market, so that the dealer now has 12 boxes a day to sell. If he has to cut the price

the quantity. In Figure 27, the first term of the elasticity formula, $\frac{dq}{dp}$ is $\frac{AB}{AP}$. The second term, $\frac{p}{q}$ is $\frac{AP}{OA}$. The formula as a whole then is $\frac{AB}{AP} \cdot \frac{AP}{OA}$. This reduces by cancellation of the two *AP*'s to $\frac{AB}{OA}$.

Since a line parallel to the base of a triangle divides the other sides propor-

tionally,
$$\frac{AB}{OA} = \frac{BP}{PC}$$
.

I am indebted to A. G. Hart for this proof, which is simpler than Marshall's. (Marshall, *Principles of Economics*, pp. 102–03, footnote 1, and the mathematical appendix, note 3, p. 839.)

"It is clear that a marginal concept is only precise when it is considered in the limiting sense, as the variations in X are made smaller and smaller. It is then to be interpreted by means of the derivative of the function which relates X and Y... Marginal revenue is thus an abstract concept only definable for continuous variations in revenue and output. But it is always approximately equal to the added revenue obtained from a small unit increase in output from the level x"—R. G. D. Allen, Mathematical Analysis for Economists, pp. 152-53. perhaps another 10 cents a box, to \$1.80, what is the marginal revenue in that case? The total revenue now is $$1.80 \times 12$, which is \$21.60. If we subtract from this the total revenue from the sale of 11 boxes, which is \$20.90, we see that the marginal revenue in this case is 70 cents.

For brevity, we say that the marginal revenue from the sale of the twelfth box of apples was 70 cents, whereas in the previous case, the marginal revenue from the sale of the eleventh box was 90 cents. But we must be careful to remember that it was not the sale of the twelfth box that brought in 70 cents, for actually that twelfth box brought in \$1.80 like all the other boxes. It was the increase in total revenue when 12 boxes were sold, over the total revenue when only 11 boxes were sold, that was 70 cents.

The marginal revenue, then, ordinarily changes as more and

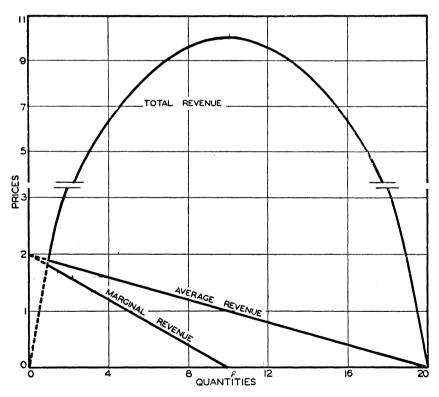


FIG. 28.—Relations between marginal, average, and total revenue. Hypothetical data.

more units are sold. The changes in total revenue, average revenue (i. e., price), and marginal revenue for various numbers of boxes of apples are shown together in Table 6. The data are plotted in Figure 28. This figure shows the simple case where the demand

TABLE 6
Apples: Total, Average, and Marginal Revenues From the Sale of
VARIOUS QUANTITIES
(Hypothetical Data)

Boxes of Apples	Price (and Average Revenuc)	Total Revenue	Marginal Revenue (Successive Differences in Totals)
1	\$1.90	\$1.90	\$1.90
		P. C.	1.70
2	1.80	3.60	1.50
3	1.70	5.10	1.30
4	1.60	6.40	1.10
5	1.50	7.50	.90
6	1.40	8.40	
7	1.30	9.10	.70
8	1.20	9.60	.50 ·
9	1.10	9.90	. 30
10	1.00	10.00	.10
			10
11	. 90	9.90	30
12	.80	9.60	50
13	.70	9.10	70
14	. 60	8.40	90
15	. 50	7.50	90

curve (or what may be called the average revenue curve, to give it a name analogous to the marginal revenue curve) is a straight line on arithmetic paper, with an average elasticity of unity.

It is clear from this figure that the slope of the marginal revenue curve is twice as steep as the slope of the average revenue curve. If the average revenue curve is a straight line, the location of the marginal revenue curve can be determined graphically without going through the calculations given above as follows: at any point on the average revenue curve, a horizontal line may be run across to the y axis. Then the marginal revenue curve cuts this line at the midpoint of the line. This holds true no matter what the elasticity of the average revenue curve may be. It also holds true for curved average revenue curves as well as straight ones, but applies in that case to the tangents to the curves.⁵

It is also clear from the chart that the values of the marginal revenue curve are positive or plus (the curve lies above the x axis) wherever the elasticity of the average revenue curve is greater than unity. (We know from previous discussion that this elasticity is greater than unity in the upper half of this particular curve.) Conversely, the values of the marginal revenue curve are negative or minus (the curve lies below the x axis) wherever the elasticity of the average revenue curve is less than unity (as it is in the lower half of this particular curve). And finally, the value of the marginal curve is zero at the point where the elasticity of the average revenue curve is unity; and at that point the total revenue reaches its maximum.

The mathematical relation between the elasticity of the average revenue curve and the values of the average and marginal revenue curves is expressed by the formula:

$$\mathbf{E} = \frac{\mathbf{AR}}{\mathbf{AR} - \mathbf{MR}}$$

In this formula,

E = Elasticity of average revenue curve

AR = Value of the average revenue curve

MR = Value of the marginal revenue curve

PLOTTING DISCRETE SERIES

When discrete quantity and price series are plotted so as to show the average, total, and marginal revenue curves, the plotter may be

⁵Care is needed in this application, for where the average revenue curve is curved, the midpoint of the horizontal line from the point of tangency to the yaxis does not lie on the marginal revenue curve. It is the line running through that point from the point where the tangent cuts the y axis, extended "southeast" until it cuts a line dropped from the point of tangency to the x axis that gives the marginal revenue at the point of tangency. This may be explained more clearly by reference to Figure 28. Obviously, the marginal revenue when 10 units are sold, at an average revenue (price) of \$1.00, is 0; that is where a line dropped from the average revenue point (\$1.00) when 10 units are sold, is cut by the line running from \$2.00 and a quantity of 0 through the midpoint of the horizontal line from the point \$1.00 at quantity 10, and the y axis.

puzzled by the fact that the marginal revenue curve apparently falls half a unit too far to the right on the chart.

This results when the curve is plotted incorrectly. The marginal revenue and average curves appear similar, but actually the two curves show not merely different things but different kinds of things.

The average revenue curve (actually, series of steps) shows the upper right-hand corners of a series of rectangles each extending to the x and y axes. The area of each rectangle shows the total revenue for each quantity and price in the series. Each figure in the scale along the bottom of the chart should be put under the mark representing the right-hand edge of each rectangle. A line drawn through the extreme point of each corner represents the average revenue curve. This curve remains the same (in the same place) for different size units of production and price.

The marginal revenue curve or series of steps, however, shows merely the tops of successive vertical bars, each one only one unit wide, showing the marginal revenue at each successive scale of production. Each figure in the scale along the bottom of the chart should be put under the center of each vertical bar. A line drawn through the center of the top of each vertical bar represents the marginal revenue curve. This line remains the same no matter how large or small the units are.

The total revenue curve is similar in kind to the average revenue curve. It should be handled in the same manner.

FORMULA FOR DETERMINING THE PRODUCTION THAT WILL BRING THE MAXIMUM TOTAL REVENUE

The size of the crop or production that will bring in the highest total revenue can be computed quickly and easily by means of a simple formula. The derivation of this formula can be visualized by remembering that if the data are plotted in index form, so that the base is 100 = the average of the series, a tangent to the demand curve would cut the x axis to the right of 100 at a point equal to the coefficient of elasticity (ignoring sign) multiplied by 100.

If, for example, the elasticity were -0.5, the tangent would cut the x axis at 150. The marginal revenue curve then would cut the x axis halfway between 0 and 150, that is, at 75. This would be the size of crop or production that would bring in the maximum total revenue.

The formula is P = (1 + E) 100

2

where P is the production that maximizes total revenue, and E is the coefficient of elasticity of demand, with the sign ignored.

BEARING UPON AGRICULTURAL POLICY

The relations between elasticity and total revenue shown above have a great deal of significance for agricultural policy.

The AAA production control program during the 1930's was designed to increase agricultural income by reducing agricultural production.

The program actually was only an acreage control program.

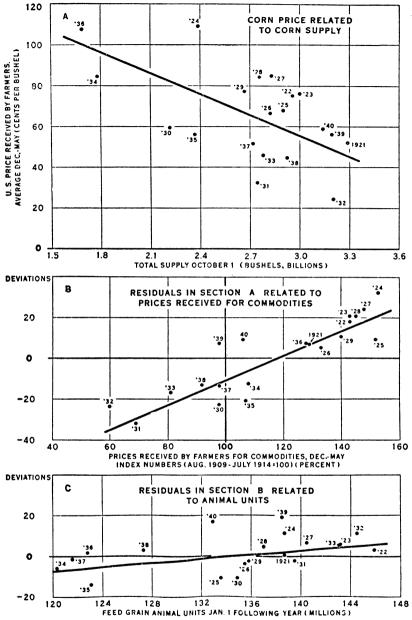
TABLE 7

FACTORS DETERMINING THE UNITED STATES AVERAGE FARM PRICE OF CORN, 1921-40

Year Beginning Month Specified	Average United States Farm Price of Corn per Bushel, December– May*	Total Corn Supply October 1†	Index Numbers of Prices Received by Farmers December- May (August 1909-July 1914 = 100) ‡	Feed Grain Animal Units January 1 of Next Year §
	(cents)	(million bushels)		(million)
1921	52.0	3,290	129	`138.7 ´
1922	74.9	2,960	143	145.9
1923	76.0	3,000	143	143.2
1924	109.2	2,390	153	138.7
1925	68.0	2,900	152	133.6
1926	66.5	2,825	133	135.5
1927	85.0	2,833	148	140.5
1928	84.3	2,757	145	137.0
1929	77.2	2,669	140	135.8
1930	59.4	2,217	98	134.9
1931	32.4	2,743	69	139.5
1932	24.2	3,202	60	144.5
1933	45.7	2,786	81	143.1
1934	84.6	1,798	108	120.3
1935	55.9	2,369	107	123.1
1936	107.6	1,687	. 128	122.8
1937	51.5	2,718	98	121.6
1938	45.0	2,925	92	127.3
1939	56.0	3,192	98	138.4
1940	58.6	3,144	106	132.9

* Agricultural Statistics 1940, XXIV, p. 55, and Crops and Markets 1935, XXIII, P. 530, both BAE, USDA.
† Feed Statistics, FdS-Sup. 2, processed, March, 1941, p. 11.
‡ Agricultural Situation (22).
§ Livestock, Meats, and Wool Market Statistics and Related Data 1940, U. S. Agr. Mktg.

Serv., processed, 1941, p. 7.



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FIG. 29.—Relations between the United States average farm price of corn December-May and (A) total United States corn supply October 1; (B) prices received by farmers December-May; and (C) number of animal units on farms January 1, 1921-22 to 1937-38.

Except in the case of cotton, it failed to reduce production below previous levels, because farmers offset the reduced acreage by recourse to production practices which increased yields. This left total production as high as before, or higher.

Even if the program had succeeded in reducing agricultural production, that would not have had much effect on agricultural income. The smaller supplies would have raised prices, but the effect of the higher price upon income would have been partially offset, completely offset, or more than offset, by the smaller supplies, depending upon the elasticities of the demand for the products concerned.

Statistical analyses have shown that the elasticity of the demand for corn in the United States, based on annual data,⁶ is about -0.65. This is shown in Figure 29, based on the data given in Table 7. The elasticity for hogs is about the same.

The general relation between hog supplies, prices, and total income, can be set forth as in Table 8. For simplicity, the figures used are percentages, with 100 representing average size. The relation

Hog Supply	Hog Price	Total Income
65	150	97
70	144	101
75	138	103
80	131	105
85	124	105
90	116	104
95	108	103
100	100	100
105	92	97
110	84	92
115	76	87
		·

TABLE 8 Relation Between Hog Supplies, Prices, and Total Income (Percentage of average)

between hog supplies and prices is shown in section A of Figure 30; the relation between hog supplies and total hog income is shown in section B of Figure 30.

Table 6 shows that a large crop of hogs is worth less than a small crop. It shows that a 110 per cent crop, for example, brings a total

^eControlling Corn and Hog Supplies and Prices, USDA Tech. Bul. No. 826, 1942, pp. 18-19, by the present author.

income only 92 per cent of average, but a 90 per cent crop brings a total income 104 per cent of average. The large crop of hogs is worth 12 per cent less than the small crop.

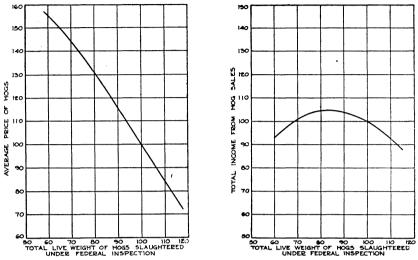


FIG. 30.—Relation between total live weight of hogs slaughtered under federal inspection, and (A) average hog prices, and (B) total revenue from hog sales.

The difference between the total values of still larger and smaller crops is still greater than this. A 115 per cent hog crop brings an 87 per cent income. This is 18 per cent less than the income from an 85 per cent crop, which is 105 per cent of average. The rise in total income with decreasing size of crop, however, stops below crop sizes of about 83 per cent. A reduction in the size of below 69 per cent of normal would reduce the total value of the crop below the value even of an average crop.

The demand curve for hogs is compared with a demand curve of constant unit elasticity at every point (which would result in a constant income no matter what the size of the crop) in Figure 31. The figure shows how the upper parts of the two curves, over the range shown, lie close together. The lower parts of the curves diverge strongly, the divergence increasing with the size of the crop. The bigger the crop, the farther does total income decline.

The conclusions given above are based upon the relations between annual data. If longer periods of time were used, the elasticity would increase and the maximum increase in total value that could be brought about by reductions in supply would decline to less than 5 per cent.

It is evident, therefore, that programs to restrict the production

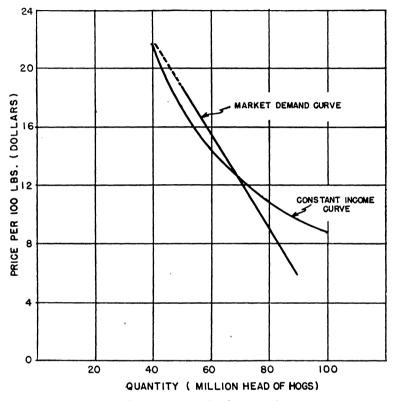


FIG. 31.-Actual market demand curve for hogs, and constant-income curve.

of crops with demand curves like those for corn and hogs could increase gross incomes only to a small extent in the short run, and probably not at all in the long run.