5

Point Elasticity, Total Revenue, Marginal Revenue

The discussion in the preceding chapter dealt only with the elasticity of the curve as a whole. But this is rather a rough-and-ready concept, for "*the* elasticity" of a curve is really a sort of average of the elasticities at different points along the curve. "*The* elasticity" we have been dealing with is often called *arc* elasticity, the elasticity of the arc of the curve. It is a sort of average elasticity. Whatever term is used (arc or average), it relates to the elasticity of the curve or arc as a whole. We come now to consider *point* elasticity, the elasticity at any point on a curve.

CURVES WITH AVERAGE ELASTICITY OF UNITY

Elasticity is a proportional concept, and the elasticity of a straight-line curve on a chart with arithmetic scales therefore is not constant from point to point along the line. It varies from point to point. This shows up most clearly in the simplest case of a straight-line curve with an average or arc elasticity of unity; such a curve is represented by the straight line marked A in section A of Figure 5.1.

This straight-line curve has an elasticity of unity at its central point where price $= 5$, and quantity $= 5$; for if $dx = dy$, as it does here when the slope of the line is 45°, then $\frac{dx}{-dy} \cdot \frac{5}{5} = -1.0$.

But at other points along the line the elasticity is not -1. At the point where price $= 6$, and quantity $= 4$, for example, the elasticity is $\frac{dx}{-dy} \cdot \frac{6}{4} = -1.5$. At the point where price $= 8$, and quantity $= 2$, the elasticity is -4.0. Conversely, at points below and to the right of the center of the line, the elasticities are less than -1.

What would a curve of constant unit elasticity at all points look like? It follows from the preceding paragraph that the slope of the line ($\frac{dy}{dx}$) at every point would have to be proportional to the relation between y and x at every point. A curve of this sort would be a rectangular hyperbola, approaching the x and y axes as asymptotes. Several constant-unit elasticity curves of this sort, lettered *B, C, D,* etc., are plotted in Section A of Figure 5.1 along with the straight-line curve *A* that has an average elasticity of unity.

This figure shows graphically how the elasticity of a straight line changes from point to point. It shows this by comparison of the straight-line curve, *A,* with the constant elasticity curves, *B, C, D,* etc., beside it. This comparison shows that the upper part of the straight-line curve is less steeply sloped than the constant unit elasticity curves; that is, it is more elastic than unity. Conversely,

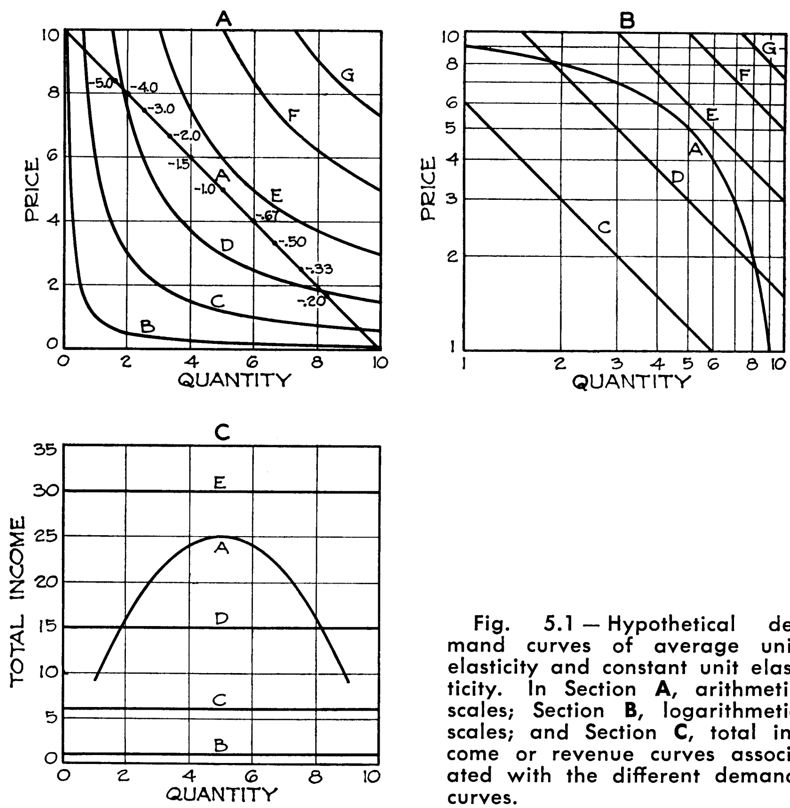

Fig. 5.1 — Hypothetical demand curves of average unit elasticity and constant unit elasticity. In Section **A**, arithmetic scales; Section **B**, logarithmetic scales; and Section **C**, total income or revenue curves associated with the different demand curves.

the lower part of the straight-line curve is less elastic than unity. The elasticity is highest at the upper end and lowest at the lower end. It is higher than unity at the top, decreases to unity at the middle, and gets less and less than unity from there on down. The elasticity at different points along the straight line is shown by the series of figures written beside the line.

The situation can be shown on double logarithmic paper, as in Section B of Figure 5.1. The constant unit elasticity curves, C, D, E, etc., shown in Section A of Figure 5.1, become straight lines with slopes of 45° on the logarithmic paper used in Section B. The straight-line curve, A, on arithmetic paper in Section A undergoes the opposite change to become a curved line, convex from above,[1] on the logarithmic paper used in Section B.

TOTAL REVENUE OR INCOME

The total revenue (that is, total income) that would be realized from the sale of different quantities of a commodity depends upon the shape or curvature of the demand curve, as well as upon its elasticity. The total revenue curve is directly related to the demand curve (which is in other words the *average* revenue curve). They can be shown on similar charts, the only difference being that in the total revenue chart the vertical scale shows total revenues instead of average revenues (prices).

The total revenue curve associated with a straight-line demand curve on arithmetic paper, with an average elasticity of unity, is shown in Section C in Figure 5.1 This shows that with this sort of a demand curve the maximum total revenue is realized from an average crop. Large crops and small crops both bring in less money than average crops. The point of highest total revenue comes at the point where the elasticity of the demand curve is unity.

The total revenue curves derived from constant unit elasticity

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[1] There are two ways of verbally describing the curvature of lines plotted on co-ordinate paper. Both of them are in common use. The one way is to describe curves as concave or convex from above (a basin with water in it is concave from above) while the other way is to describe them as concave or convex to the origin.

There are objections to both systems. The objection to the "from above" reference is that it cannot be applied to curves whose ends lie on the same vertical line. The objection to the "origin" reference is that it cannot be applied to curves that go through the origin. This objection is perhaps more important than the other, because some important economic curves necessarily start from the origin—total revenue curves, positive sloping curves of unit elasticity, etc. In addition, most mathematicians (although not R. G. D. Allen) and some economists use the reference "from above"—Joan Robinson in England, and Tintner, Waugh, and Thomsen in the United States, to name only a few. We are accordingly using the reference "from above" in the present work.

curves like *B, C, D,* etc., are horizontal straight lines. If the elasticity is unity at all points, the total revenue remains constant, whatever the size of the crop, as shown for the curves *B, C, D,* and *E* in Section C of Figure 5.1.

Concave demand curves with an average elasticity of unity, but less curved on arithmetic paper than the constant elasticity curves shown in Section A of Figure 5.1, undergo an interesting transformation when plotted on logarithmic paper. Their curvature is reversed. They are concave on arithmetic paper, but they become convex on logarithmic paper. The point of highest total revenue appears (in their case as in the case of straight-line curves on arithmetic paper) at the central point of the curve where the point elasticity is unity.

Concave demand curves with an average elasticity of unity, but more curved on arithmetic paper than constant-elasticity curves, lose some of their curvature when plotted on logarithmic paper. But they retain their concavity. Accordingly, with this kind of demand curve, the *minimum* total revenue is realized from the sale of an average crop. Large crops and small crops both bring in more money than an average crop. The point where the elasticity of the demand curve is unity is the point of lowest total revenue.

ELASTICITIES OTHER THAN UNITY

Demand curves whose elasticity is constant but higher or lower than unity are straight lines on logarithmic scales, like the curves *C, D,* etc., in Section B of Figure 5.1, but their slopes are other than 45°. The slopes of the inelastic curves are steeper than 45°, and those of the elastic curves, flatter than 45°.

The total revenue curves associated with these constant (but not unit) elasticity demand curves are sloping curved lines, not horizontal straight lines like the total revenue curves associated with constant unit elasticity curves. They are curved lines. The total revenue curves for constant but less than unit elasticity demand curves are rectangular hyperbolas like the curves of constant elasticity shown earlier in Section A of Figure 5.1. And like those curves, they would be straight lines with a negative slope on double logarithmic paper. A demand curve with constant elasticity of —0.5 is shown on arithmetic and logarithmic paper, together with the total income or revenue curve based upon it, in the four sections of Figure 5.2.

The total revenue curves for constant but more than unit elasticity demand curves are also curved, but they are parabolas,

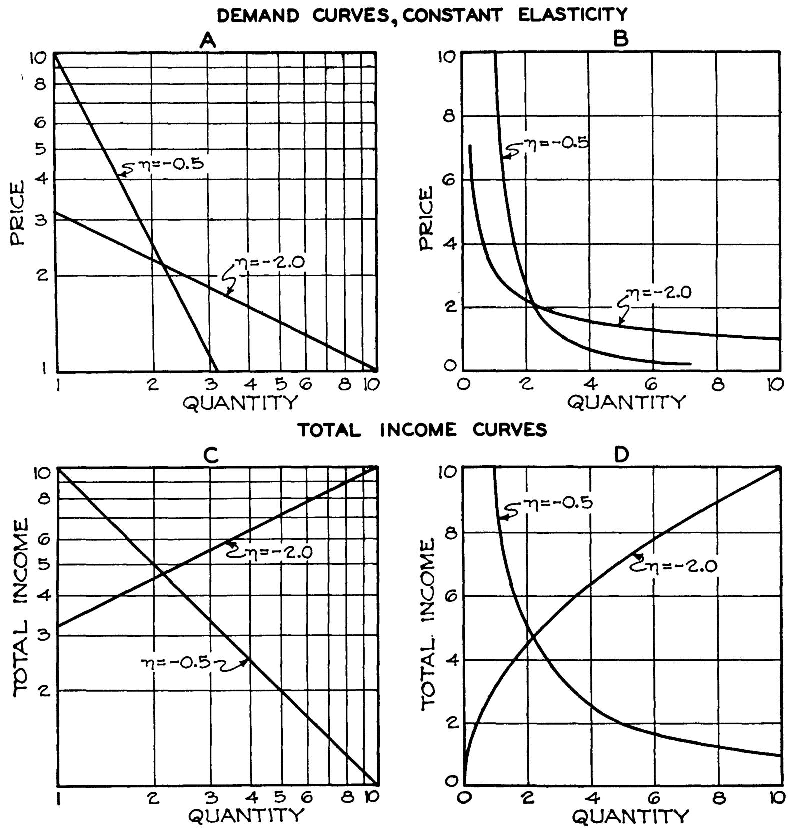

Fig. 5.2 — Sections **A** and **B**, hypothetical constant elasticity demand curves; Sections **C** and **D**, total income (revenue) curves associated with them.

with apex at the origin of the x and y axes. They, too, are straight lines on double logarithmic paper; but they have a positive slope. A demand curve with a constant elasticity of —2.0, and the total revenue curve based upon it, is shown also on arithmetic and logarithmic paper in Figure 5.2.

The elasticity figures given beside the total revenue curves in the lower sections of this chart are only identification-elasticities, showing in each case the elasticity of the demand curve from which the total revenue curve was derived, not the elasticity of the total revenue curve itself. That elasticity can be figured mathematically from the elasticity of the demand curve. The mathematical relationship between the elasticity of the demand curve and that of its associated total revenue curve is comparatively simple. If $E =$ the elasticity of the total revenue curve, defined in an analogous manner to the Marshallian definition of the elasticity of the demand curve (i.e., as $\frac{dq}{d(pq)} \cdot \frac{pq}{q}$ or $\frac{dq}{q} / \frac{d(pq)}{pq}$) and e = the elasticity of the demand curve, the relationship[2] is $E = \frac{e}{e+1}$

This formula may be solved for e in terms of E, giving the elasticity of the demand curve in terms of the elasticity of the total revenue curve, as follows: $e = \frac{E}{1-E}$

THE MEASUREMENT OF POINT ELASTICITY

Point elasticity can be measured mathematically by the use of the same formula that was used for average or arc elasticity in the preceding chapter. If the demand curve is a straight line, the formula is merely the original Marshallian formula, with the p and q at the particular point used in place of the *average* p and *average* q over the range used when average elasticity is computed. If the

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[2] This relationship may be derived as follows:

$$E = \frac{dq}{q} / \frac{d(pq)}{pq} = 1/\frac{d(pq)}{pdq} = 1/\frac{pdq+qdp}{pdq}$$
$$= 1/\left(1 + \frac{qdp}{pdq}\right) = 1/\left(1 + \frac{1}{e}\right) = \frac{e}{e+1}$$

It is interesting to observe that if E and e are both defined as reciprocals of the usual definitions, that is, as $\frac{d(pq)}{dq} \cdot \frac{q}{pq}$ and $\frac{dp}{dq} \cdot \frac{q}{p}$ respectively, then the relation between E and e is very simple. It is

$$E = \frac{d(pq)}{dq} \cdot \frac{q}{pq} = \frac{d(pq)}{dq} \cdot \frac{1}{p} = \frac{qdp+pdq}{pdq} = \frac{qdp}{pdq} + 1 = e + 1.$$

(*See* R. G. D. Allen, *Mathematical Analysis for Economists,* Macmillan, London, 1938, p. 252.) That is, E as thus defined is always greater by 1 than e as thus defined.

If e is defined as usual (as $\frac{dq}{dp} \cdot \frac{p}{q}$) and E defined as above (as $\frac{d(pq)}{dq} \cdot \frac{q}{pq}$) then $E = \frac{1}{e} + 1$.

I am indebted to Gerhard Tintner and Adolf Kozlik for this footnote.

curve is not a straight line, then a tangent must be drawn to it at the point where the elasticity is to be measured. The $\frac{dq}{dp}$ is then computed from the tangent.

Point elasticity can also be measured graphically. The way to do it is shown in Figure 5.3. If the demand curve is a straight line and the elasticity at the point p is to be found, that can be done by laying a ruler along the demand curve and measuring the two distances (1) from p to the point where the ruler cuts the y axis (pc in the diagram) and (2) from p to the point where the ruler cuts the x axis (pb in the diagram). The latter distance divided by the former (that is, $\frac{pb}{pc}$) then gives the elasticity at the point p.

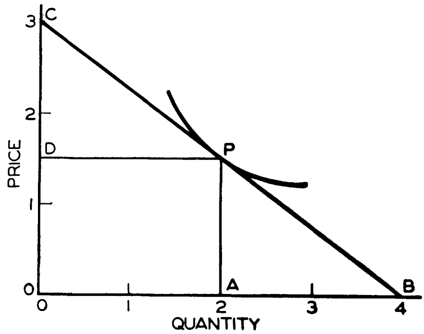

Fig. 5.3 — Diagram for the measurement of point elasticity.

If the demand curve is a curved line, the procedure is the same, but the ruler is laid tangent to the curve at the point where the elasticity is to be measured. The ratio $\frac{pb}{pc}$ then gives the elasticity at that point.[3] It can then be shown that the elasticity at p is also given by $\frac{ab}{oa}$.

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[3] The proof of these relations is simple. The elasticity, $\frac{dq}{dp} \cdot \frac{p}{q}$, is the change in quantity divided by the change in price, multiplied by the price divided by the quantity. In Figure 5.3, the first term of the elasticity formula, $\frac{dq}{dp}$ is $\frac{AB}{AP}$. The second term, $\frac{p}{q}$ is $\frac{AP}{OA}$. The formula as a whole then is $\frac{AB}{AP} \cdot \frac{AP}{OA}$. This reduces by cancellation of the two AP's to $\frac{AB}{OA}$.

Since a line parallel to the base of a triangle divides the other sides proportionally, $\frac{AB}{OA} = \frac{BP}{PC}$.

I am indebted to A. G. Hart for this proof, which is simpler than Marshall's. (Marshall, *Principles of Economics*, pp. 102–3, footnote 1, and the mathematical appendix, note 3, p. 839.)

MARGINAL REVENUE

The preceding sections have shown the relation between elasticity and total income or revenue. We turn now to a third concept, marginal revenue.

The concept or definition of marginal revenue is perfectly clear-cut. The total revenue (or total income, which means the same thing) is the total revenue from the sale of a given amount of the product, say x bushels. It is computed by multiplying x bushels by the price at which that number of bushels can be sold. This total revenue may be compared with the total revenue from the sale of $x + 1$ bushels; this is computed by multiplying $x + 1$ bushels by the price at which that number of bushels can be sold. The difference between the two total revenues is the marginal revenue.[4]

For example, a dealer may be able to sell 10 boxes of apples a day for $2.00 a box. His total revenue from the sale of apples then is $20.00. Suppose now that more apples come on the market; he now has 11 boxes a day to sell. He has to cut the price to move them all. He cannot merely cut the price of the eleventh box; the buyers of the 10 boxes would object; so he has to cut the price of all the boxes of apples. If he has to cut the price to $1.90 per box, his total revenue then is $1.90 × 11, which is $20.90. What is the marginal revenue, then? It is the difference between $20.00 and $20.90; it is 90 cents.

Suppose then that still more apples come on the market, so that the dealer now has 12 boxes a day to sell. If he has to cut the price perhaps another 10 cents a box, to $1.80, what is the marginal revenue in that case? The total revenue now is $1.80 × 12, which is $21.60. If we subtract from this the total revenue from the sale of 11 boxes, which is $20.90, we see that the marginal revenue in this case is 70 cents.

For brevity, we say that the marginal revenue from the sale of the twelfth box of apples was 70 cents, whereas in the previous case, the marginal revenue from the sale of the eleventh box was 90 cents. But we must be careful to remember that it was not the sale of the twelfth box that brought in 70 cents, for actually that

[4] R. G. D. Allen, *Mathematical Analysis for Economists*, St. Martins, 1962, pp. 152–53: "It is clear that a marginal concept is only precise when it is considered in the limiting sense, as the variations in X are made smaller and smaller. It is then to be interpreted by means of the derivative of the function which relates X and Y. . . . Marginal revenue is thus an abstract concept only definable for continuous variations in revenue and output. But it is always approximately equal to the added revenue obtained from a small unit increase in output from the level x."

twelfth box brought in $1.80 like all the other boxes. It was *the increase in total revenue when 12 boxes were sold, over the total revenue when only 11 boxes were sold, that was 70 cents.*

The marginal revenue, then, ordinarily changes as more and more units are sold. The changes in total revenue, average revenue (i.e., price), and marginal revenue for various numbers of boxes of apples are shown together in Table 5.1. The data are plotted in Figure 5.4. This figure shows the simple case where the demand curve (or what may be called the average revenue curve, to give it a name analogous to the marginal revenue curve) is a straight line on arithmetic paper, with an average elasticity of unity.

It is clear from this figure that the slope of the marginal revenue

TABLE 5.1
APPLES: TOTAL, AVERAGE, AND MARGINAL REVENUES FROM THE SALE OF VARIOUS QUANTITIES
(Hypothetical Data)

Boxes of Apples	Price (and Average Revenue)	Total Revenue	Marginal Revenue (Successive Differences in Totals)
			$1.90
1	$1.90	$1.90	
			1.70
2	1.80	3.60	
			1.50
3	1.70	5.10	
			1.30
4	1.60	6.40	
			1.10
5	1.50	7.50	
			.90
6	1.40	8.40	
			.70
7	1.30	9.10	
			.50
8	1.20	9.60	
			.30
9	1.10	9.90	
			.10
10	1.00	10.00	
			−.10
11	.90	9.90	
			−.30
12	.80	9.60	
			−.50
13	.70	9.10	
			−.70
14	60	8.40	
			−.90
15	.50	7.50	

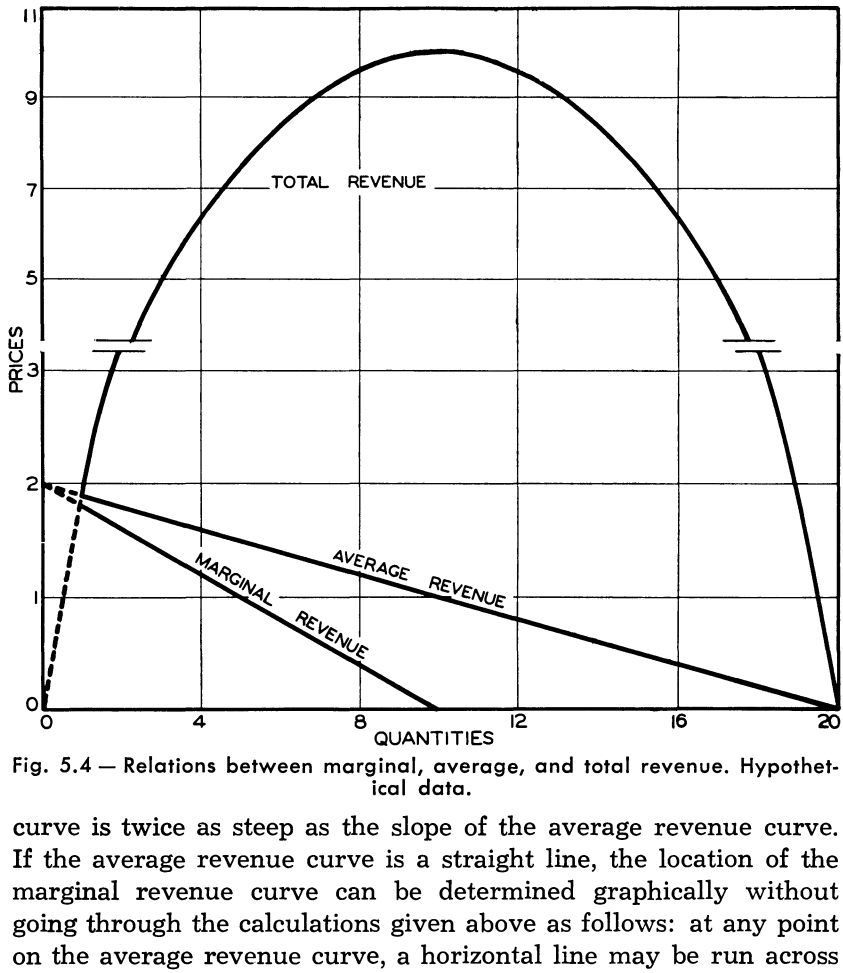

Fig. 5.4 — Relations between marginal, average, and total revenue. Hypothetical data.

curve is twice as steep as the slope of the average revenue curve. If the average revenue curve is a straight line, the location of the marginal revenue curve can be determined graphically without going through the calculations given above as follows: at any point on the average revenue curve, a horizontal line may be run across to the y axis. Then the marginal revenue curve cuts this line at the midpoint of the line. This holds true no matter what the elasticity of the average revenue curve may be. It also holds true for curved average revenue curves as well as straight ones, but applies in that case to the tangents to the curves.[5]

It is also clear from the chart that the values of the marginal rev-

[5] Care is needed in this application, for where the average revenue curve is curved, the midpoint of the horizontal line from the point of tangency to the y axis does not lie on the marginal revenue curve. It is the line running through that point from the point where the tangent cuts the y axis, extended "southeast" until it cuts a line dropped from the point of tangency to the x axis that gives the marginal revenue at the point of tangency. This may be explained more clearly by reference to Figure 5.4. Obviously, the marginal revenue when 10 units are sold, at an average revenue (price) of $1.00, is 0; that is where a line dropped from the average revenue point ($1.00) when 10 units are sold, is cut by the line running from $2.00 and a quantity of 0 through the midpoint of the horizontal line from the point $1.00 at quantity 10, and the y axis.

enue curve are positive or plus (the curve lies above the x axis) wherever the elasticity of the average revenue curve is greater than unity. (We know from previous discussion that this elasticity is greater than unity in the upper half of this particular curve.) Conversely, the values of the marginal revenue curve are negative or minus (the curve lies below the x axis) wherever the elasticity of the average revenue curve is less than unity (as it is in the lower half of this particular curve). And finally, the value of the marginal curve is zero at the point where the elasticity of the average revenue curve is unity; and at that point the total revenue reaches its maximum.

The mathematical relation between the elasticity of the average revenue curve and the values of the average and marginal revenue curves is expressed by the formula:

$$e = -\frac{AR}{AR - MR}$$

In this formula,

e = Elasticity of average revenue
AR = Value of the average revenue
MR = Value of the marginal revenue

R. H. Leftwich points out that the fact that the marginal revenue curve bisects the horizontal price line is only a mathematical happenstance. The important relation logically is that the space under the MR line is equal to the space under the horizontal price line. This equality exists when the vertical distance between the MR and AR lines (equal to one price unit at a quantity of 10 in Figure 5.4, for example) equals the vertical distance from the price on the y axis to the point on the y axis where the MR and AR curves originate.[6]

PLOTTING DISCRETE SERIES

When discrete quantity and price series are plotted so as to show the average, total, and marginal revenue curves, the plotter may be puzzled by the fact that the marginal revenue curve apparently falls half a unit too far to the right on the chart.

This results when the curve is plotted incorrectly. The marginal revenue and average curves appear similar, but actually the two curves show not merely different things but different kinds of things.

The average revenue curve (actually, series of steps) shows the upper right-hand corners of a series of rectangles each extending to the x and y axes. The area of each rectangle shows the total

[6] R. H. Leftwich, *The Price System and Resource Allocation*, Holt, 1961, Appendix I to Chap. X, pp. 225–30.

revenue for each quantity and price in the series. Each figure in the scale along the bottom of the chart should be put under the mark representing the right-hand edge of each rectangle. A line drawn through the extreme point of each corner represents the average revenue curve. This curve remains the same (in the same place) for different size units of production and price.

The marginal revenue curve or series of steps, however, shows merely the tops of successive vertical bars, each one only one unit wide, showing the marginal revenue at each successive scale of production. Each figure in the scale along the bottom of the chart should be put under the center of each vertical bar. A line drawn through the center of the top of each vertical bar represents the marginal revenue curve. This line remains the same no matter how large or small the units are.

The total revenue curve is similar in kind to the average revenue curve. It should be handled in the same manner.

FORMULA FOR DETERMINING THE PRODUCTION THAT WILL BRING THE MAXIMUM TOTAL REVENUE

The size of the crop or production that will bring in the highest total revenue can be computed quickly and easily by means of a simple formula. The derivation of this formula can be visualized by remembering that if the data are plotted in index form, so that the base is 100 = the average of the series, a tangent to the demand curve would cut the x axis to the right of 100 at a point equal to the coefficient of elasticity (ignoring sign) multiplied by 100.

If, for example, the elasticity were —0.5, the tangent would cut the x axis at 150. The marginal revenue curve then would cut the x axis halfway between 0 and 150, that is, at 75. This would be the size of crop or production that would bring in the maximum total revenue.

The formula is $P = \dfrac{(1 + e)\ 100}{2}$

where P is the production that maximizes total revenue, and e is the coefficient of elasticity of demand, with the sign ignored.

BEARING UPON AGRICULTURAL POLICY

The relations between elasticity and total revenue shown above have a great deal of significance for agricultural policy.

The AAA production control program during the 1930's was designed to increase agricultural income by reducing agricultural production.

The program actually was only an acreage control program. Except in the case of cotton, it failed to reduce production below previous levels, because farmers offset the reduced acreage by recourse to production practices which increased yields. This left total production as high as before, or higher.

Even if the program had succeeded in reducing agricultural production, that would not have had much effect on agricultural income. The smaller supplies would have raised prices, but the effect of the higher price upon income would have been partially offset, completely offset, or more than offset, by the smaller supplies, depending upon the elasticities of the demand for the products concerned.

Statistical analyses have shown that the elasticity of the demand for corn in the United States, based on annual data, is about —0.65. This is shown in Figure 8.3 at the end of Chapter 8. The elasticity for hogs is about the same. Analyses of the data since World War II indicate lower elasticities for the postwar period. Whether this decrease in elasticity is temporary or permanent is not known.

The general relation between hog supplies, prices, and total income, can be set forth as in Table 5.2. For simplicity, the figures used are percentages, with 100 representing average size. The relation between hog supplies and prices is shown in Section A of Figure 5.5; the relation between hog supplies and total hog income is shown in Section B of Figure 5.5.

Table 5.2 shows that a large crop of hogs is worth less than a small crop. It shows that a 110 per cent crop, for example, brings a total

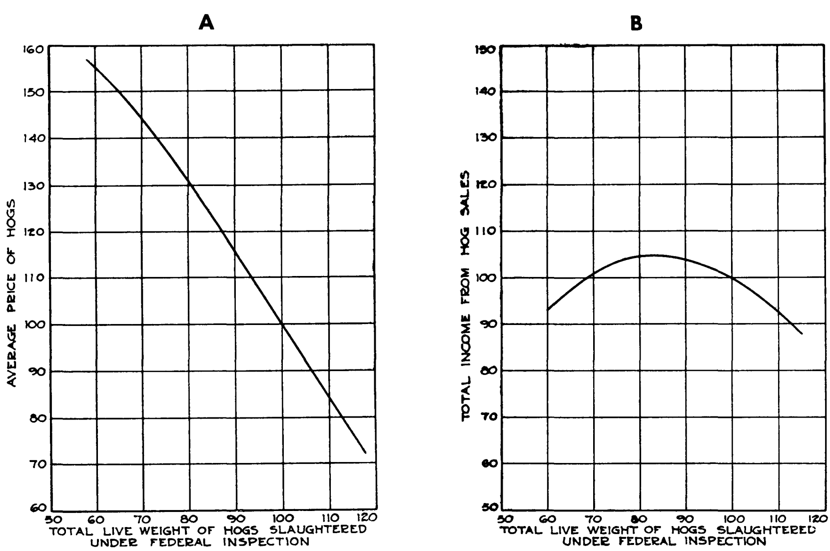

Fig. 5.5 — Relation between total live weight of hogs slaughtered under federal inspection: Section A, average hog prices; Section B, total revenue from hog sales.

TABLE 5.2

RELATION BETWEEN HOG SUPPLIES, PRICES, AND TOTAL INCOME*

(Percentage of average)

Hog Supply	Hog Price	Total Income
65	150	97
70	144	101
75	138	103
80	131	105
85	124	105
90	116	104
95	108	103
100	100	100
105	92	97
110	84	92
115	76	87

* This table is based on data in Table 9.2 and the accompanying discussion.

income only 92 per cent of average, but a 90 per cent crop brings a total income 104 per cent of average. The large crop of hogs is worth 12 per cent less than the small crop.

The difference between the total values of still larger and smaller crops is still greater than this. A 115 per cent hog crop brings an 87 per cent income. This is 18 per cent less than the income from an 85 per cent crop, which is 105 per cent of average. The rise in total income with decreasing size of crop, however, stops below crop sizes of about 83 per cent. A reduction in the size of below 69 per cent of normal would reduce the total value of the crop below the value even of an average crop.

The demand curve for hogs is compared with a demand curve of constant unit elasticity at every point (which would result in a constant income no matter what the size of the crop) in Figure 5.6. The figure shows how the upper parts of the two curves, over the range shown, lie close together. The lower parts of the curves diverge strongly, the divergence increasing with the size of the crop. The bigger the crop, the farther does total income decline.

The conclusions given above are based upon the relations between annual data. If longer periods of time were used, the elasticity would increase and the maximum increase in total value that could be brought about by reductions in supply would decline to less than 5 per cent.[7]

[7] E. J. Working in his bulletin "Demand for Meat," Institute of Meat Packing, Univ. of Chicago, 1954, marshals statistical evidence to show that "the long-run elasticity of the demand for pork is greater than the short-run elasticity" (p. 78). He uses logarithmic functions as well as linear (straight-line) functions; the former give demand curves with a slight concave curvature, which would be closer to the constant income curve shown in Figure 5.6 than the straight-line market demand curve shown in the figure.

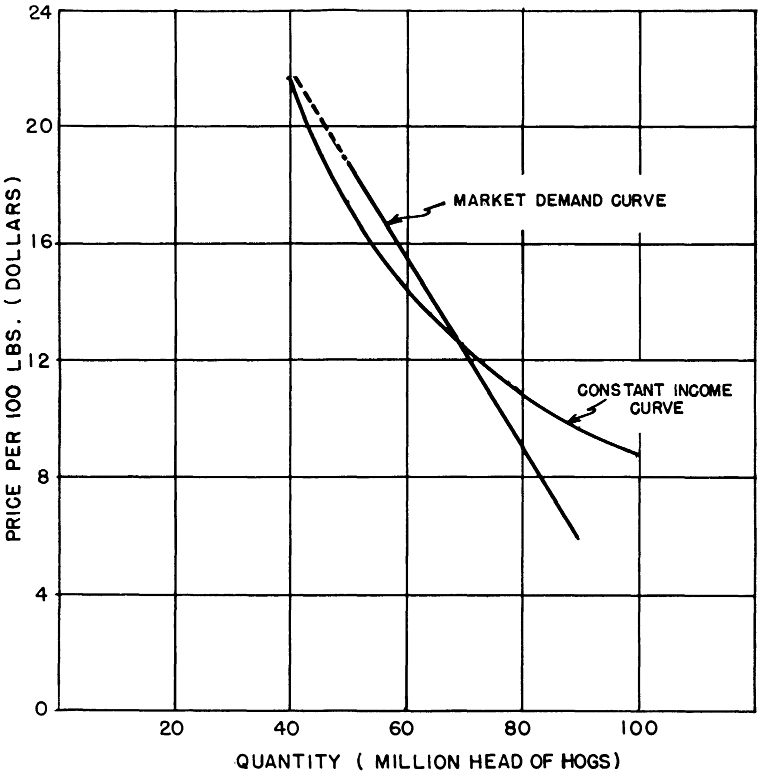

Fig. 5.6 — Actual market demand curve for hogs, and constant-income curve.

It is evident, therefore, that programs to restrict the production of crops with demand curves like those for corn and hogs could increase gross incomes only to a small extent in the short run, and probably not at all in the long run.