Quasicrystal Tilings in 3-Dimensions and their Empires

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Abstract

The cut-and-project method for computing quasicrystals is a robust algorithm which provides a mathematical framework for more detailed analysis of the tilings they generate. The method is characterized by a lattice $\Lambda \subset \mathbb{E}^N$ and its projections onto an affine subspace $\pi : \Lambda \to \mathbb{E}_\parallel \simeq \mathbb{R}^n$. The cut-window $\mathcal{W} \subset \mathbb{E}_\perp$ inside the orthogonal complement of $\mathbb{E}_\parallel$ provides a filter for determining which points are incorporated into a particular tiling, $\mathcal{T} \subset \mathbb{E}_\parallel$: a point $\lambda_\parallel$ is included in $\mathcal{T}$ if and only if $\lambda_\perp$ falls within the cut-window $\mathcal{W}$. The cut-window contains regions corresponding to individual tiles, a particular tile is attached to a point $\lambda_\parallel$ if and only if $\lambda_\perp$ falls within that tile's corresponding region inside $\mathcal{W}$. Taking the intersections of overlapping regions decomposes the cut-window into sectors which correspond to individual vertex configurations. Computing the relative volumes of these regions gives analytical values for the vertex frequencies. We also present an algorithm for defining a region in the cut-window which corresponds to the forced tiles, local configuration, and the empire given an arbitrary set of initial tiles. We focus on tilings of $\mathbb{R}^3$ and present constructions and analysis for the Ammann tiling (projection of $\Lambda = \mathbb{Z}^6 \to \mathbb{R}^3$) as well as a quasicrystal with 36 vertex types ($\Lambda = D_6 \to \mathbb{R}^3$) as studied extensively by Kramer.

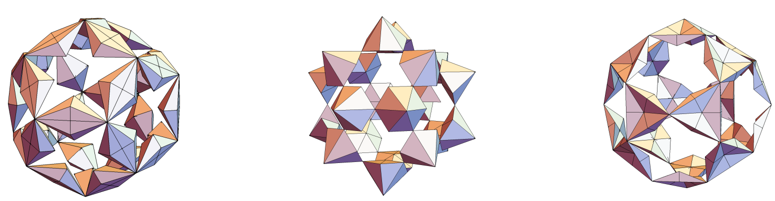

Figure 1: Various sectors in the cut-window corresponding to three different vertex types in a quasicrystal defined by a cut-and-projection of the D_6 lattice onto $\mathbb{R}^3$.