Tilings with Singularity and Spiral Tilings Obtained from Archimedean Tilings

Jonathan V. Caalim, Manuel Joseph C. Loquias, Angelica R. Sta. Ana

Institute of Mathematics, College of Science, University of the Philippines Diliman,

1101 Quezon City, Philippines

arstaana1@up.edu.ph

A *singular point* $\mathcal{P}$ of a plane tiling is a point in the plane where every circular disk about $\mathcal{P}$ meets infinitely many tiles of the tiling. Tilings with a singular point were obtained in [1] by applying the conformal map

$$\varphi_\alpha(z) = \exp\left(\frac{2\pi i}{\alpha} z\right) \tag{1}$$

to the three regular Archimedean tilings for certain values of α. The symmetry group of each resulting tiling with singularity is isomorphic either to a finite cyclic or dihedral group. Moreover, for each positive integer n, it is possible to construct a tiling with singularity that has rotation symmetry of order n by choosing a suitable α.

This contribution aims to generate tilings with singularity by applying the same φ_α to the remaining eight semiregular Archimedean tilings. We obtain sufficient conditions for α so that the image of a semiregular Archimedean tiling under the map φ_α is a tiling with a singular point. In addition, we also identify the symmetry groups of the resulting tilings with singularity. For instance, if ω denotes $\exp(2\pi i/3)$, then the images of the 3.6.3.6 tiling under φ_α (where $\alpha = -10+2\omega$) and the 3.4.3.3.4 tiling under φ_α (where $\alpha = -4+4\omega$) are both tilings with singularity at the origin as shown in Fig. 1 and Fig. 2, respectively. Observe that the tiling in Fig. 1 has symmetry group isomorphic to the cyclic group C_2 while the tiling in Fig. 2 has symmetry group isomorphic to the dihedral group D_4.

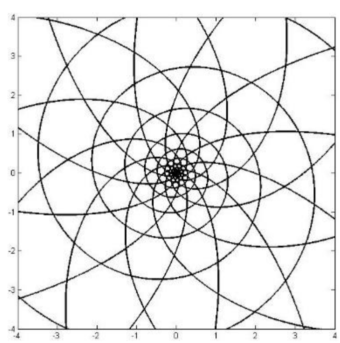

Figure 1. φ_α(3.6.3.6) where $\alpha = -10+2\omega$.

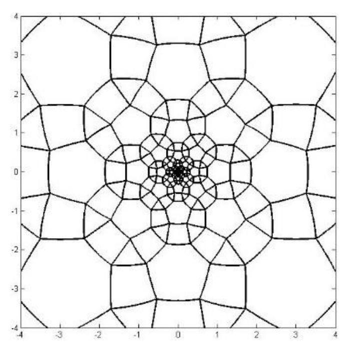

Figure 2. φ_α(3.4.3.3.4) where $\alpha = -4+4\omega$.

Loosely speaking, tilings which give the viewer a psychological “spiral effect” are said to be *spiral*. In this case, the tiling in Fig. 1 is spiral. On the other hand, the tiling in Fig. 2 contains no “spiral effect”. There are several efforts in the literature to give a suitable mathematical definition for spiral tilings. Very recently, Klaassen [2] proposed several criteria for a tiling to be called spiral. A tiling $\Im$ is said to be *spiral-like* if there is a partition of $\Im$ into classes (called *arms*) such that for each arm, there exists a simple unbounded non-intersecting curve (called *thread*) meeting each tile in an arm exactly once while spinning infinitely often around a point. Two tiles that belong to the same arm A of a spiral-like tiling are referred to as *direct neighbors* if their intersection is cut by the thread of A or contains more than a finite number of points. A spiral-like tiling with exactly one singular point is then called a *spiral tiling* if whenever a pair of tiles T_1 and T_2 that are direct neighbors are mapped by rotation, translation, and scaling onto another pair of tiles T_3 and T_4, then tiles T_3 and T_4 are also direct neighbors within an arm. We test these criteria by looking at whether the tiling $\varphi_\alpha(\Im)$ with singularity obtained from some Archimedean tiling $\Im$ is a spiral tiling according to the definition given in [2].

To illustrate, the image of the square tiling 4^4 under the map φ_α, where $\alpha = 6$, gives no "spiral effect" as shown in Fig. 3. However, this tiling can be partitioned into arms that satisfy the conditions of a spiral tiling. One such partition is given in Fig. 4 where one arm corresponds to one color and the threads are denoted by the broken logarithmic spirals.

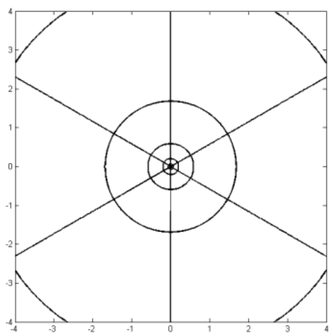

Figure 3. $\varphi_\alpha(4^4)$ where $\alpha = 6$.

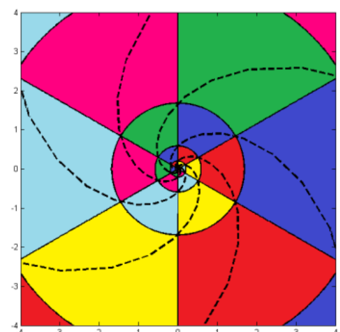

Figure 4. A partition of $\varphi_\alpha(4^4)$ where $\alpha = 6$.

1. I. F. Evidente, R. P. Felix, R., and M. J. C. Loquias, *Acta Cryst.*, **A71**, (2015), 583.

2. B. Klaassen, *Math. Mag.*, **90**, (2017), 26.